Detailed Derivations of *Whole-Body Real-Time Motion Planning for Multicopters*

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Abstract

This technical report can be viewed as a complementary part for *Whole-Body Real-Time Motion Planning for Multicopters*, from which the readers are assumed to be directed. We will derive the derivatives of attitude penalty in details.

1 Preliminaries

We denote the body attitude $\boldsymbol{R}_b = \begin{bmatrix}\boldsymbol{r}_{1b} & \boldsymbol{r}_{2b} & \boldsymbol{r}_{3b}\end{bmatrix}$ of a quadrotor as:

$$\boldsymbol{r}_{1b} \coloneqq \frac{\boldsymbol{s}}{AB} \tag{1a}$$

$$\boldsymbol{r}_{2b} \coloneqq \frac{\boldsymbol{k}}{B} \tag{1b}$$

$$\boldsymbol{r}_{3b} \coloneqq \frac{\boldsymbol{t}}{A} \tag{1c}$$

$A, B \in \mathbb{R}$, $\boldsymbol{s}, \boldsymbol{k}, \boldsymbol{t} \in \mathbb{R}^3$. It has been shown in the paper that all of them are functions of differentially flat outputs $\boldsymbol{\sigma} = [p_x, p_y, p_z, \psi]^{\mathrm{T}}$:

$$A \coloneqq \sqrt{\ddot{p_x}^2 + \ddot{p_y}^2 + (\ddot{p_z} + g)^2} \tag{2a}$$

$$B \coloneqq \sqrt{(\ddot{p_z} + g)^2 + (\ddot{p_x}\sin\psi - \ddot{p_y}\cos\psi)^2} \tag{2b}$$

$$\boldsymbol{s} \coloneqq \begin{bmatrix} (\ddot{p_z} + g)^2\cos\psi + \ddot{p_y}^2\cos\psi - \ddot{p_x}\ddot{p_y}\sin\psi \\ (\ddot{p_z} + g)^2\sin\psi + \ddot{p_x}^2\sin\psi - \ddot{p_x}\ddot{p_y}\cos\psi \\ -(\ddot{p_z} + g)(\ddot{p_x}\cos\psi + \ddot{p_y}\sin\psi) \end{bmatrix} \tag{2c}$$

$$\boldsymbol{k} \coloneqq \begin{bmatrix} -(\ddot{p_z} + g)\sin\psi \\ (\ddot{p_z} + g)\cos\psi \\ \ddot{p_x}\sin\psi - \ddot{p_y}\cos\psi \end{bmatrix} \tag{2d}$$

$$\boldsymbol{t} \coloneqq \begin{bmatrix} \ddot{p_x} \\ \ddot{p_y} \\ \ddot{p_z} + g \end{bmatrix} \tag{2e}$$

Our trajectory is composed of several pieces, each piece i is represented by the following:

$$\boldsymbol{p}_i(t) \coloneqq \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{\beta}_0(t), t \in [0, T_i] \tag{3a}$$
$$\boldsymbol{v}_i(t) = \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{\beta}_1(t) \tag{3b}$$
$$\boldsymbol{a}_i(t) = \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{\beta}_2(t) = \begin{bmatrix}\ddot{p_x} & \ddot{p_y} & \ddot{p_z}\end{bmatrix}^{\mathrm{T}} \tag{3c}$$
$$\boldsymbol{j}_i(t) = \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{\beta}_3(t) \tag{3d}$$
$$\boldsymbol{\beta}_0(t) \coloneqq [1, t, t^2, \cdots, t^N]^{\mathrm{T}} \tag{3e}$$
$$\boldsymbol{\beta}_j(t) = \boldsymbol{\beta}_0^{(j)}(t), j \in \{1, 2, 3\} \tag{3f}$$

with $\boldsymbol{c}_i \in \mathbb{R}^{(N+1)\times 3}$ to be the polynomial coefficient matrix and $T_i \in \mathbb{R}$ is the traverse time of the i-th trajectory.

2 Derivation of Derivatives of Attitude Penalty

$$\mathcal{G}_{att}^{k,v}(\boldsymbol{c}_i, T_i, \hat{t}) = (\boldsymbol{n}_i^k)^{\mathrm{T}}(\boldsymbol{p}_i(\hat{t}\cdot T_i) + \boldsymbol{R}_b^i(\hat{t}\cdot T_i)\tilde{\boldsymbol{q}}_{\boldsymbol{v}} - \boldsymbol{o}_i^k) \tag{4}$$

represents how far away the point $\tilde{\boldsymbol{q}}_{\boldsymbol{v}} \in \mathbb{R}^3$ is outside the k-th hyperplane of i-th polyhedron. We are saying outside because the normal vector $\boldsymbol{n}_i^k$ is pointing inwards the polyhedron by our assumption and $\boldsymbol{o}_i^k$ is one point on the plane. $\tilde{\boldsymbol{q}}_{\boldsymbol{v}}$ is typically some constant offset from the CoM of drone. $\hat{t} \in [0, 1]$ is a normalized timestamp. If $\mathcal{G}_{att}^{k,v}(\boldsymbol{c}_i, T_i, \hat{t}) > 0$, then $\tilde{\boldsymbol{q}}_{\boldsymbol{v}}$ is outside the i-th polyhedron.

As been pointed out in the paper, we are interested in finding the gradient $\frac{\partial \mathcal{G}_{att}^{k,v}}{\partial \boldsymbol{c}_i}$ and $\frac{\partial \mathcal{G}_{att}^{k,v}}{\partial T_i}$. For the notation to be further simplified, denote

$$\boldsymbol{n} = \begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}^{\mathrm{T}} = \boldsymbol{n}_i^k \tag{5a}$$
$$\boldsymbol{o} = \boldsymbol{o}_i^k \tag{5b}$$
$$\boldsymbol{R}_b^i(\hat{t}\cdot T_i) = \begin{bmatrix}\boldsymbol{r}_{1b} & \boldsymbol{r}_{2b} & \boldsymbol{r}_{3b}\end{bmatrix} \tag{5c}$$
$$\tilde{\boldsymbol{q}}_v = \begin{bmatrix}\tilde{q}_x & \tilde{q}_y & \tilde{q}_z\end{bmatrix}^{\mathrm{T}} \tag{5d}$$
$$m = \mathcal{G}_{att}^{k,v}(\boldsymbol{c}_i, T_i, \hat{t}) \tag{5e}$$

Then the single penalty term has expression:

$$m = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{p}_i(\hat{t}\cdot T_i) - \boldsymbol{n}^{\mathrm{T}}\boldsymbol{o} + \tilde{q}_x\boldsymbol{n}^{\mathrm{T}}\boldsymbol{r}_{1b} + \tilde{q}_y\boldsymbol{n}^{\mathrm{T}}\boldsymbol{r}_{2b} + \tilde{q}_z\boldsymbol{n}^{\mathrm{T}}\boldsymbol{r}_{3b} \tag{6}$$

Combining $\boldsymbol{r}_{1b}$ from (1a),$\boldsymbol{r}_{2b}$ from (1b) and $\boldsymbol{r}_{3b}$ from (1c) with (6), the derivative of m with respect to $\boldsymbol{c}_i$ and T_i are:

$$\begin{aligned}\frac{\partial m}{\partial \boldsymbol{c}_i} = {} & \boldsymbol{\beta}_0(\hat{t}\cdot T_i)\boldsymbol{n}^{\mathrm{T}} + \frac{\tilde{q}_x}{AB}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}}{\partial \boldsymbol{c}_i} + \tilde{q}_x\boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}\frac{\partial \frac{1}{AB}}{\partial \boldsymbol{c}_i} \\ & + \frac{\tilde{q}_y}{B}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}}{\partial \boldsymbol{c}_i} + \tilde{q}_y\boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}\frac{\partial \frac{1}{B}}{\partial \boldsymbol{c}_i} \\ & + \frac{\tilde{q}_z}{A}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}}{\partial \boldsymbol{c}_i} + \tilde{q}_z\boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}\frac{\partial \frac{1}{A}}{\partial \boldsymbol{c}_i}\end{aligned} \tag{7a}$$

$$
\begin{aligned}
\frac{\partial m}{\partial T_i} &= \frac{\partial m}{\partial(\hat{t}\cdot T_i)}\frac{\partial(\hat{t}\cdot T_i)}{\partial T_i} \\
&= \hat{t}\cdot\Big(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v}_i(\hat{t}\cdot T_i) + \frac{\tilde{q}_x}{AB}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}}{\partial(\hat{t}\cdot T_i)} + \tilde{q}_x\boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}\frac{\partial\frac{1}{AB}}{\partial(\hat{t}\cdot T_i)} \\
&+ \frac{\tilde{q}_y}{B}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}}{\partial(\hat{t}\cdot T_i)} + \tilde{q}_y\boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}\frac{\partial\frac{1}{B}}{\partial(\hat{t}\cdot T_i)} \\
&+ \frac{\tilde{q}_z}{A}\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}}{\partial(\hat{t}\cdot T_i)} + \tilde{q}_z\boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}\frac{\partial\frac{1}{A}}{\partial(\hat{t}\cdot T_i)}\Big)
\end{aligned} \tag{7b}
$$

with $\boldsymbol{\beta}_0(\cdot)$ defined in (3e) and $\boldsymbol{v}_i(\cdot)$ defined in (3b).

Define temporal variables $\alpha, \beta \in \mathbb{R}$

$$\alpha \coloneqq \ddot{p_x}^2 + \ddot{p_y}^2 + (\ddot{p_z} + g)^2 \tag{8a}$$

$$\beta \coloneqq (\ddot{p_z} + g)^2 + (\ddot{p_x}\sin\psi - \ddot{p_y}\cos\psi)^2 \tag{8b}$$

which leads to $\frac{1}{A} = \alpha^{-\frac{1}{2}}, \frac{1}{B} = \beta^{-\frac{1}{2}}$ with A and B defined in (2a) and (2b) respectively. Vectors $\boldsymbol{j}_i(\cdot)$ and $\boldsymbol{\beta}_2(\cdot)$ appearing below are defined in (3d) and (3f) respectively.

Now calculate the partial derivatives $\frac{\partial\frac{1}{A}}{\partial\boldsymbol{c}_i}$, $\frac{\partial\frac{1}{A}}{\partial(\hat{t}\cdot T_i)}$ assuming that yaw angle remains zero and $\boldsymbol{t}$ as defined in (2e):

$$\frac{\partial\frac{1}{A}}{\partial\boldsymbol{c}_i} = \frac{\partial\alpha^{-\frac{1}{2}}}{\partial\alpha}\frac{\partial\alpha}{\partial\boldsymbol{c}_i} = -\frac{1}{2}A^{-3}\frac{\partial\alpha}{\partial\boldsymbol{c}_i} = -A^{-3}\boldsymbol{\beta}_2(\hat{t}\cdot T_i)\boldsymbol{t}^{\mathrm{T}} \tag{9a}$$

$$\frac{\partial\frac{1}{A}}{\partial(\hat{t}\cdot T_i)} = \frac{\partial\alpha^{-\frac{1}{2}}}{\partial\alpha}\frac{\partial\alpha}{\partial(\hat{t}\cdot T_i)} = -\frac{1}{2}A^{-3}\frac{\partial\alpha}{\partial(\hat{t}\cdot T_i)} = -A^{-3}\boldsymbol{j}_i(\hat{t}\cdot T_i)^{\mathrm{T}}\boldsymbol{t} \tag{9b}$$

Similar result can be obtained for $\frac{\partial\frac{1}{B}}{\partial\boldsymbol{c}_i}$, $\frac{\partial\frac{1}{B}}{\partial(\hat{t}\cdot T_i)}$ under same assumptions with $\boldsymbol{k}$ defined in (2d) and $\tilde{\boldsymbol{k}}$ newly defined by (10c):

$$\frac{\partial\frac{1}{B}}{\partial\boldsymbol{c}_i} = \frac{\partial\beta^{-\frac{1}{2}}}{\partial\beta}\frac{\partial\beta}{\partial\boldsymbol{c}_i} = -\frac{1}{2}B^{-3}\frac{\partial\beta}{\partial\boldsymbol{c}_i} = -B^{-3}\boldsymbol{\beta}_2(\hat{t}\cdot T_i)\tilde{\boldsymbol{k}}^{\mathrm{T}} \tag{10a}$$

$$\frac{\partial\frac{1}{B}}{\partial(\hat{t}\cdot T_i)} = \frac{\partial\beta^{-\frac{1}{2}}}{\partial\beta}\frac{\partial\beta}{\partial(\hat{t}\cdot T_i)} = -\frac{1}{2}B^{-3}\frac{\partial\beta}{\partial(\hat{t}\cdot T_i)} = -B^{-3}\boldsymbol{j}_i(\hat{t}\cdot T_i)^{\mathrm{T}}\tilde{\boldsymbol{k}} \tag{10b}$$

$$\tilde{\boldsymbol{k}} \coloneqq \begin{bmatrix} (\ddot{p_x}\sin\psi - \ddot{p_y}\cos\psi)\sin\psi \\ (\ddot{p_x}\sin\psi - \ddot{p_y}\cos\psi)(-\cos\psi) \\ \ddot{p_z} + g \end{bmatrix} \tag{10c}$$

Apply chain rule to results from (9) and (10), the terms $\frac{\partial\frac{1}{AB}}{\partial\boldsymbol{c}_i}$, $\frac{\partial\frac{1}{AB}}{\partial(\hat{t}\cdot T_i)}$ are simply:

$$\frac{\partial\frac{1}{AB}}{\partial\boldsymbol{c}_i} = \frac{1}{A}\frac{\partial\frac{1}{B}}{\partial\boldsymbol{c}_i} + \frac{1}{B}\frac{\partial\frac{1}{A}}{\partial\boldsymbol{c}_i} \tag{11a}$$

$$\frac{\partial \frac{1}{AB}}{\partial(\hat{t}\cdot T_i)} = \frac{1}{A}\frac{\partial \frac{1}{B}}{\partial(\hat{t}\cdot T_i)} + \frac{1}{B}\frac{\partial \frac{1}{A}}{\partial(\hat{t}\cdot T_i)} \tag{11b}$$

The derivative for term $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}$ is easy to compute:

$$\boldsymbol{n}^{\mathrm{T}}\boldsymbol{t} = n_1\ddot{p_x} + n_2\ddot{p_y} + n_3(\ddot{p_z} + g) \tag{12a}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}}{\partial \boldsymbol{c}_i} = \boldsymbol{\beta}_2(\hat{t}\cdot T_i)\boldsymbol{n}^{\mathrm{T}} \tag{12b}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{t}}{\partial(\hat{t}\cdot T_i)} = \boldsymbol{j}_i(\hat{t}\cdot T_i)^{\mathrm{T}}\boldsymbol{n} \tag{12c}$$

The terms $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}$ and $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}$ are rather complicated and requires expanding the normal vector itself. Note that $\boldsymbol{s}$ is defined in (2c). We also introduce newly defined $\tilde{\boldsymbol{n}_{\boldsymbol{k}}}$ and $\tilde{\boldsymbol{n}_{\boldsymbol{s}}}$ in (13d) and (14d).

$$\boldsymbol{n}^{\mathrm{T}}\boldsymbol{k} = \ddot{p_x}n_3\sin\psi - \ddot{p_y}n_3\cos\psi + (\ddot{p_z} + g)(n_2\cos\psi - n_1\sin\psi) \tag{13a}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}}{\partial \boldsymbol{c}_i} = \boldsymbol{\beta}_2(\hat{t}\cdot T_i)\tilde{\boldsymbol{n}_{\boldsymbol{k}}}^{\mathrm{T}} \tag{13b}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{k}}{\partial(\hat{t}\cdot T_i)} = \boldsymbol{j}_i(\hat{t}\cdot T_i)^{\mathrm{T}}\tilde{\boldsymbol{n}_{\boldsymbol{k}}} \tag{13c}$$

$$\tilde{\boldsymbol{n}_{\boldsymbol{k}}} := \begin{bmatrix} n_3\sin\psi \\ -n_3\cos\psi \\ n_2\cos\psi - n_1\sin\psi \end{bmatrix} \tag{13d}$$

$$\begin{aligned}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{s} = &(\ddot{p_z} + g)^2(n_1\cos\psi + n_2\sin\psi) + \ddot{p_y}^2 n_1\cos\psi + \ddot{p_x}n_2\sin\psi \\ &- \ddot{p_x}\ddot{p_y}(n_1\sin\psi + n_2\cos\psi) - (\ddot{p_z} + g)\ddot{p_x}n_3\cos\psi - (\ddot{p_z} + g)\ddot{p_y}n_3\sin\psi\end{aligned} \tag{14a}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}}{\partial \boldsymbol{c}_i} = \boldsymbol{\beta}_2(\hat{t}\cdot T_i)\tilde{\boldsymbol{n}_{\boldsymbol{s}}}^{\mathrm{T}} \tag{14b}$$

$$\frac{\partial \boldsymbol{n}^{\mathrm{T}}\boldsymbol{s}}{\partial(\hat{t}\cdot T_i)} = \boldsymbol{j}_i(\hat{t}\cdot T_i)^{\mathrm{T}}\tilde{\boldsymbol{n}_{\boldsymbol{s}}} \tag{14c}$$

$$\tilde{\boldsymbol{n}_{\boldsymbol{s}}} := \begin{bmatrix} 2\ddot{p_x}n_2\cos\psi - \ddot{p_y}(n_1\sin\psi + n_2\cos\psi) - (\ddot{p_z} + g)n_3\cos\psi \\ 2(\ddot{p_z} + g)(n_1\cos\psi + n_2\sin\psi) - \ddot{p_x}n_3\cos\psi - \ddot{p_y}n_3\sin\psi \\ 2(\ddot{p_z} + g)(n_1\cos\psi + n_2\sin\psi) - \ddot{p_x}n_3\cos\psi - \ddot{p_y}n_3\sin\psi \end{bmatrix} \tag{14d}$$

By combining all individual components above into (7a) and (7b), we get the single gradient of $\frac{\partial \mathcal{G}_{att}^{k,v}}{\partial \boldsymbol{c}_i}$ and $\frac{\partial \mathcal{G}_{att}^{k,v}}{\partial T_i}$.